CLASIFICATION

- linear — u and all derivatives are linear
- non-linear
  - quasilinear — derivatives of principal order linear ~~with~~ coeff's being dependent on lower order derivatives
  - semilinear — quasilinear and coeff's depend only on x (not u)
  - fully nonlinear — otherwise

$$\underbrace{a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy}}_{\equiv \mathcal{L}u} + a_1 u_x + a_2 u_y + a_0 u = 0$$

↳ characteristic curves: $y(x)$

satisfy: $a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\frac{dy}{dx} + a_{22} = 0$

$$D := a_{11}a_{22} - a_{12}^2$$

eliptic if $D > 0$ → $\mathcal{L}u = \pm(u_{XX} + u_{YY})$ ☆ Laplace eq.
0 real char. curves

~ all evals have same sign

parabolic if $D = 0$ → $\mathcal{L}u = \pm u_{XX}$ ☆ Heat eq.
1 char. curve

~ one or more evals vanish

hyperbolic if $D < 0$ → $\mathcal{L}u = u_{XX} - u_{YY}$ ☆ Wave eq.
2 char. curves through each point

~ all but 1 eval have same sign, none vanish

1. Find D → type of eq.
2. Complete square for $\mathcal{L}$ to be in canonical form (above)
3. Identify ∂_X, ∂_Y from completed square
4. Generate matrix B : $\begin{pmatrix}\partial_X \\ \partial_Y\end{pmatrix} = B\begin{pmatrix}\partial_x \\ \partial_y\end{pmatrix}$
5. Solve the ODE in terms of X, Y
6. Generate B^T from B → use it to go back to original variables: $\begin{pmatrix}x \\ y\end{pmatrix} = B^T\begin{pmatrix}X \\ Y\end{pmatrix}$ → find then X, Y in terms of x, y and substitute into solution

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FOURIER SERIES

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(k\frac{\pi}{l}x\right) + b_k \sin\left(k\frac{\pi}{l}x\right)$$

$$f(x) = \sum_{k=1}^{\infty} c_k e^{-ikx\frac{\pi}{l}} \qquad a_k = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(k\frac{\pi}{l}x\right)dx$$

$$\frac{1}{2l}\int_{-l}^{l} f(x) e^{-ikx\frac{\pi}{l}} dx \qquad b_k = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(k\frac{\pi}{l}x\right)dx$$

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f\bar{g}\, dx \qquad \rightarrow \quad a_k = \langle f, \cos(kx)\rangle, \quad b_k = \langle f, \sin(kx)\rangle$$

$\langle \cos, \sin\rangle = 0$

$\langle \overset{\sin}{\cos}(kx), \overset{\sin}{\cos}(lx)\rangle = \delta_{kl}$

$$S_n = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)$$

periodic extension: $\tilde{f}(x+2\pi) = \tilde{f}(x)$ and $\tilde{f}(x) = f(x) \quad -\pi < x \leq \pi$

mag. of jumps: $\beta_k = f(x_k^+) - f(x_k^-)$ | and $\tilde{f}(\pi) = \tilde{f}(-\pi) = \frac{1}{2}\left[f(\pi) + f(-\pi)\right]$

CONVERGENCE

Thm: f - 2π periodic piecewise C^1, $x \in \mathbb{R}$ $\Rightarrow$ FS converges pointwise to $\frac{1}{2}\left[\tilde{f}(x^+) + \tilde{f}(x^-)\right]$

Thm: $n \in \mathbb{N}_0$, if FS coeffs satisfy $\sum_{k=-\infty}^{\infty} |k|^n |c_k| < \infty$, $c_k = \langle f, e^{ikx}\rangle$, then FS converges uniformly to ~~...~~ $\tilde{f}(x) \in C^n$ ($\tilde{f}$: 2π periodic extension of f)

(+ uniform convergence of derivative $\leq n$)

Thm: $I \subset \mathbb{R}$ ~~...~~ for $\forall k \in \mathbb{N}$, $|u_k(x)| \leq m_k$ $\forall x \in I$ with $m_k \geq 0$

~~and~~ if $\sum_{k=1}^{\infty} m_k < \infty$, then ~~f~~ $\sum_{k=1}^{\infty} u_k(x) = f(x)$ converges uniformly to $f(x)$

~~...~~ Bessel's inequality: $\|f\|^2 \geq \sum_{k=1}^{n} |c_k|^2$

Plancherel's ~~eq~~: $\|f\|^2 = \sum_{k=1}^{\infty} |c_k|^2 = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2 + b_k^2$

Parseval: $\langle f, g\rangle = \sum_{k=1}^{\infty} c_k \bar{d}_k = ?$

$\|f\|^2 = \|f - S_n + S_n\|^2 = \|f - S_n\|^2 + \|S_n\|^2$ by orthogonality $\rightarrow 0 \leq \|f - S_n\|^2 = \|f\|^2 - \|S_n\|^2 = \|f\|^2 - \sum^{n} |c_k|^2$

FOURIER TRANSFORM

$$f(x) = \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}}_{\mathcal{F}^{-1}} \underbrace{\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-iku} du \right)}_{\mathcal{F}} dk$$

$$\mathcal{F}\left(\frac{\partial^n v}{\partial x^n}\right) = (ik)^n \mathcal{F}(v)$$

convolutions: $f * g = \int_{-\infty}^{\infty} f(u) g(x-u) du$

$$\Rightarrow \widehat{f*g} = \sqrt{2\pi}\, \hat{f} \cdot \hat{g}, \quad \check{f} * \check{g} = \sqrt{2\pi}\, \check{(f \cdot g)}$$

$$\hat{\delta}(x-\xi) = \frac{1}{\sqrt{2\pi}} e^{-i\xi k}$$

$$\mathcal{F}[e^{-ax^2}] = \frac{1}{\sqrt{2a}} e^{-k^2/4a}$$

$$\mathcal{F}[e^{-a|x|}] = \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2+k^2} \quad \Rightarrow \quad \mathcal{F}^{-1}\left[\frac{2a}{a^2+k^2}\right] = \sqrt{2\pi}\, e^{-a|x|}$$

$$\mathcal{F}\left[\begin{cases} 1 & -T \le t \le T \\ 0 & \text{otherwise} \end{cases}\right] = \sqrt{\frac{2}{\pi}} \frac{\sin(\omega T)}{\omega} \quad \rightarrow \quad \mathcal{F}^{-1}\left[\frac{2\sin(\omega T)}{\omega}\right] = \sqrt{2\pi} \begin{cases} 1 & -T \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}} \longrightarrow \mathcal{F}^{-1}[1] = \sqrt{2\pi}\, \delta(x)$$

CONVERGENCE (FS)

Pointwise: $\forall x$, $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n \ge N$: $|f_n(x) - f(x)| < \varepsilon$

Uniform: $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n \ge N$, $\forall x$: $|f_n(x) - f(x)| < \varepsilon$

↳ preserves continuity

in norm: $\|S_n - f\| = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} |S_n - f|^2 dx} \xrightarrow[n \to \infty]{} 0$

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pointwise: if $\lim_{n \to \infty} v_n(x)$ exists

uniform: if $\sup |f_n(x) - f(x)| \xrightarrow[n \to \infty]{} 0$

- not uniform: find $x(n)$ for which $v_n(x)$ is indep. of n $\Rightarrow \not\xrightarrow[n\to\infty]{} 0$
- ✓ uniform: bound sup by something indep. of x $\xrightarrow[n\to\infty]{} 0$
- or find sup using derivs

TRANSPORT EQUATION

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad u(0,x) = f(x)$$

use: $\xi = x - ct \to v(t,\xi)$

$\Rightarrow \frac{\partial v}{\partial t} = 0 \to v(t,\xi) = v(0,x) = v(x)$

$v(0,x) = u(0,x) = f(x)$

$v(t,\xi) = u(t,x) = f(x-ct)$

$$\underbrace{\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}}_{\frac{\partial v}{\partial t} = -av} = -au$$

$\Rightarrow v(t,\xi) = g(\xi)e^{-at}$

~~$v(t,\xi) = v(t, x-ct)$~~

$v(0,\xi) = v(0,x) = u(0,x) = f(x) = g(\xi)$

$u(t,x) = e^{-at} f(x-ct)$

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = 0, \quad u(0,x) = f(x)$$

- solutions are const. along char. curves
- horizontal or strictly monotone
- don't cross

$h(t) = u(t, x(t)) \Rightarrow \frac{dh}{dt} = \frac{\partial u}{\partial t} + \underset{\text{"}c(x)\text{"}}{\frac{dx}{dt}}\frac{\partial u}{\partial x} = 0$

$\beta(x) = \int \frac{dx}{c(x)} = t + K \Rightarrow \xi = \beta(x) - t$ char. variable

$\beta^{-1}(t+K) = x(t)$

$\downarrow$ char. curve (slope c)

$u(t,x) = f \circ \beta^{-1}(\beta(x) - t) = f \circ \beta^{-1}(\xi)$

more general

If $c(t)$:

char. eq. $\frac{dx}{dt} = c$ → solve to form: $\beta(x) = \alpha(t) + K$

→ $\xi = K = \beta(x) - \alpha(t)$

→ function of ξ is sol.

→ at $t = 0$ (or t_0): $\xi = \beta(x) \to x = \beta^{-1}(\xi)$

→ $u(0,x) = f(x) = f(x(\xi)) = f(\beta^{-1}(\xi))$

→ $u(t,x) = f \circ \beta^{-1}(\xi(t,x))$

WAWE EQUATION

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$\xi = x - ct$, $\eta = x + ct \to \partial_\xi \partial_\eta u = 0$

$\to u(\xi,\eta) = F(\xi) + G(\eta)$

$u(0,x) = f(x)$, $u_t(0,x) = g(x)$

D'alembert formula

$$u(t,x) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(s,y)\,dy\,ds$$

↳ for $u_{tt} = c^2 u_{xx} + F$

Thm: Any sol. can be written as $u(t,x) = F(x-ct) + G(x+ct)$

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} \left(\frac{\partial u}{\partial t}\right)^2 + c^2\left(\frac{\partial u}{\partial x}\right)^2 dx$$

Separation of variables $u = v(x)w(t) \to w'' - \lambda w = 0$, $v'' - \frac{\lambda}{c^2}v = 0$

$\lambda < 0$: sin/cos
$\lambda = 0$: $1, x$; $1, t$
$\lambda > 0$: sinh/cosh

for bounded dom. (periodic sol.)

$$u(t,x) = \sum_{n=0} \left[a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right]\left[c_n \cos\left(\omega_n \frac{x}{c}\right) + d_n \sin\left(\omega_n \frac{x}{c}\right)\right]$$

Dirichlet (odd extensions of f, g to show that d'Alembert satisfies boundary conditions)

$u(t,0) = 0 = u(t,l)$

$b_n = \frac{2}{l}\int_0^l f \sin\left(\frac{n\pi x}{l}\right)dx$, $d_n = \frac{2}{n\pi c}\int_0^l g \sin\left(\frac{n\pi x}{l}\right)dx$

$$u(t,x) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right) + d_n \sin\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

even extend

$u_x(t,0) = 0 = u_x(t,l)$

↳ Neumann

$\frac{1}{l}\int_0^l f\,dx$ | $\frac{1}{l}\int_0^l g\,dx$ | $\frac{2}{l}\int_0^l f\cos\left(\frac{n\pi x}{l}\right)dx$ | $\frac{2}{n\pi c}\int_0^l g\cos\left(\frac{n\pi x}{l}\right)dx$

$$u(t,x) = a_0 + c_0 t + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi ct}{l}\right)\cos\left(\frac{n\pi x}{l}\right) + c_n \sin\left(\frac{n\pi ct}{l}\right)\cos\left(\frac{n\pi x}{l}\right)$$

HEAT EQUATION

$u_t = \gamma u_{xx}$ or $u_t = \Delta u$ $\quad u(0,x) = f(x)$

- Sol. can be written as $u(x) = \sum_{j=1}^{\infty} \alpha_j v_j(x)$

$\alpha_j = \int_\Omega f(x)\, \overline{v_j}(x)\, dx \,,\; \int_\Omega v_j \cdot \overline{v_k}\, dx = \delta_{jk}$ $\quad$ ⇒ eigenfunctions forming a complete ~~ortho~~ orthonormal set

more concretely for heat eq:

sep vars $\quad u(x,t) = \sum_{m=1}^{\infty} A_m v_m(x) e^{-\lambda_m t} \,,\quad A_m = \int_\Omega f(x)\, \overline{v_m(x)}\, dx$

Solving:

$u(t,x) = w(t) v(x) \quad \rightarrow -\lambda = \frac{w'(t)}{w(t)} = \gamma \frac{v''(x)}{v(x)}$

$\Rightarrow w(t) = \quad \rightarrow \quad w'(t) = -\lambda w(t) \quad \rightarrow \quad w(t) = e^{-\lambda t}$

$v''(x) = -\frac{\lambda}{\gamma} v(x) \quad \rightarrow \quad v''(x) + \frac{\lambda}{\gamma} v = 0$

Dirichlet

Try $v(x) = a\cos(\omega x) + b\sin(\omega x) \,,\quad \omega = \sqrt{\frac{\lambda}{\gamma}}$

→ impose boundary conditions

$\rightarrow \quad u(t,x) = \sum_{n=1}^{\infty} b_n e^{-\gamma \frac{n^2\pi^2}{\ell^2} t} \sin\left(\frac{n\pi x}{\ell}\right)$, $\quad$ uniqueness from boundary conditions or $E(t) = \int_{-\infty}^{\infty} |w(t,x)|^2 dx$

with $b_n = \frac{2}{\ell} \int_0^\ell f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$

- $|f(x)|$ integrable $\rightarrow |b_n|$ bounded $\rightarrow |b_n| \leq \frac{2}{\ell}\int_0^\ell |f(x)|\, dx = M$ $\quad \Rightarrow |u(t,x)| \leq M \sum_n e^{-\gamma \frac{n^2\pi^2}{\ell^2} t}$ (geo series)

as $t \to \infty$, $|u(t,x)| \to 0$ ⇒ dissipative

- smoothing

Inhomogeneous boundary conditions: $\quad u(t,0) = \alpha$, $\quad u(t,\ell) = \beta$

$\tilde{u} = u - u_1$ $\quad$ (general, solves homogeneous problem; inhomog; particular solving inhomog bndry)

$\rightarrow u_1(x) = \alpha + \left(\frac{\beta - \alpha}{\ell}\right) x$ $\quad$ particular sol

$\rightarrow \quad u(t,x) = \alpha + \left(\frac{\beta-\alpha}{\ell}\right) x + \sum_{n=1}^{\infty} b_n e^{-\gamma \frac{n^2\pi^2}{\ell^2} t} \sin\left(\frac{n\pi x}{\ell}\right)$

Maximum principle for parabolic

the max (and min) value of $u(x,t)$ is assumed, either initially $(t=0)$ or ~~on~~ at $x=0$ ∨ $x=\ell$

~~Proof~~ $\quad \Rightarrow w = u_1 - u_2$ is by ~~boundary~~ equivalence of boundary conditions 0 at boundaries (and. t=0)

⇒ 0 everywhere

on whole line:

$u(t,x) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, dy$

$u_t = k u_{xx} \,,\; u(x,0) = \phi(x)$

LAPLACE $f(x,y)=0$ & POISSON's EQUATIONS

↳ solved by harmonic functions

$$\Delta u = u_{xx} + u_{yy} = -f(x,y) \quad \to \quad \left.\begin{aligned} v''-\lambda v &= 0 \\ w''+\lambda w &= 0 \end{aligned}\right\} \begin{aligned} &\text{are sin/cos} \\ &\text{are sinh/cosh} \end{aligned}$$

★ rectangle: $\Omega = [0,a]\times[0,b]$, $u(x,0)=f(x)$, $u(x,b)=u(0,y)=u(a,y)=0$

$$\Rightarrow u(x,y) = \sum \frac{b_n}{\sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right) + \underbrace{Ax+By+Cxy+D}_{\text{try to match first to boundary conditions}}$$

$$b_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

★ circle: $\Delta u = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

$$\Rightarrow u(r,\theta) = \frac{a_0}{2} + b_0\log(r) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left[a_n r^n\cos(n\theta) + b_n r^n \sin(n\theta)\right]$$

- exterior: only $n \le 1$
- interior: only $n \ge 1$

★ For Dirichlet on interior: $\frac{a_n}{R^n} = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos(n\theta)d\theta$

$$\frac{b_n}{R^n} = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\sin(n\theta)d\theta$$

Poisson's formula:

in polar: $u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\frac{(1-r^2)\,d\phi}{1+r^2-2r\cos(\theta-\phi)}$

cartesian: $u(x) = \frac{a^2-|x|^2}{2\pi a}\int_{|y|=a}\frac{g(y)}{|x-y|^2}dy$ → volume $= \frac{a^2-|x|^2}{n\omega_n a}\int_{|y|=a}\frac{g(y)}{|x-y|^2}dy$ Kernel $K(x,y)$ initial data

⇓ harmonic $x<|a|$ and $\lim_{\substack{x\to y \\ |x|<a}}(u(x)) = g(y)$, $|y|=a$

u $\in C^2(\Omega) \cap C^1(\partial\overline{\Omega})$:

- subharmonic: $-\Delta u \le 0$ ⟹ weak max.: $\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x)$
- harmonic: $-\Delta u = 0$ – both
- superharmonic: $-\Delta u \ge 0$ – weak min.: $\min_{x\in\overline{\Omega}} u(x) = \min_{x\in\partial\Omega} u(x)$ primer

in Ω bounded open connected

$-\Delta u_1 = f_1$, $-\Delta u_2 = f_2$ Ω
$u_1 = g_1$, $u_2 = g_2$ $\partial\Omega$
$f_1 < f_2$, $g_1 \le g_2 \Rightarrow u_1 \le u_2$

Mean Value Formula: $u(0) = \overline{u}_a$ ← average of u over $\partial B_a(0)$; a – radius of ball

GREEN'S FUNCTION I

$w = v_2 - v_1$

$$\boxed{E = \int |\nabla w|^2 \, dx}$$

$= \langle v_2, f\rangle - \langle v_1, f\rangle$

Fundamental solution = Green's fun on unbounded

for Laplace: $\Delta V_F(x) = 0$

$$\int_{\mathbb{R}^n} V_F(x)\, \Delta\phi(x)\, dx = \phi(0) \quad \text{for } \forall \phi \in C_c^\infty(\mathbb{R}^n)$$

c: compact support

$$\left[\text{Green's Thm: } \int_\Omega (f\Delta g - g\Delta f) = \int_{\partial\Omega}\left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) dx\right]$$

For 3D, spherically symmetric: $V'' + \frac{2}{r}V' = 0$

$\Rightarrow V = \underbrace{C_1}_{\text{choose as } 0} + \underbrace{\frac{C_2}{r}}_{\text{solves everywhere but at } 0}$

$\Omega = \{x;\ \varepsilon \le |x| \le M\}$

$$C_2 \lim_{\varepsilon \downarrow 0} \int_{\varepsilon \le r \le M} \frac{1}{r}\Delta\phi\, dx - \int_{\varepsilon \le r \le M} \Delta\left(\frac{1}{r}\right)\phi\, dx =$$

$$= C_2 \lim_{\varepsilon \downarrow 0} \int_{r=M} \frac{\partial\phi}{\partial r}\frac{1}{r}\, dx - \int_{r=M} \phi\frac{\partial}{\partial r}\frac{1}{r}\, dx - \int_{r=\varepsilon} \frac{\partial\phi}{\partial r}\frac{1}{r}\, dx + \int_{r=\varepsilon} \phi\frac{\partial}{\partial r}\frac{1}{r}\, dx =$$

first two terms: 0 by compact support

$$\le \frac{1}{\varepsilon} \underbrace{\sup\left|\frac{\partial\phi}{\partial r}\right|}_{C = \text{const.}} |B_\varepsilon| = \frac{1}{\varepsilon} C\, 4\pi\varepsilon^2 = 4\pi C\varepsilon \xrightarrow{\varepsilon\to 0} 0$$

$$\left(= -\int_{r=\varepsilon} \phi \frac{1}{r^2} = -\frac{1}{\varepsilon^2}\int_{r=\varepsilon} \phi\, dx = -4\pi\bar{\phi}_\varepsilon \xrightarrow{\varepsilon\to 0} \phi(0) \right)$$

$$= 0 + 0 = 0 + C_2(-4\pi\phi(0)) \equiv \phi(0) \Rightarrow C_2 = -\frac{1}{4\pi} \Rightarrow V_F = -\frac{1}{4\pi r}$$

Def using $\delta(x)$: $\Delta V_F(x) = \delta(x)$ $\quad \int \Delta V_F\, \phi\, dx = \int V_F\, \Delta\phi\, dx = \int \delta(x)\phi(x) = \phi(0)$

For Poisson: $V(x) = \int_{\mathbb{R}^n} \underbrace{V_F}_{\text{fund. sol.}}(x-y) f(y)\, dy = \int_{\mathbb{R}^n} \frac{-f(y)}{4\pi(x-y)}\, dy$ if 3D symmetric spherically

any other sol. to Poisson can be rearranged to this form

proof: $\Delta V = f$
$\Delta V_F = \delta(x - x_0)$ $\quad \Delta V_F = \delta(x - x_0)$

compact support $\Rightarrow \int_{\mathbb{R}^n} \tilde{V}\Delta V - V\Delta\tilde{V}\, dx$

any function $V \in C_0^2(\mathbb{R}^3)$ satisfies:

$$V(x) = -\int_{\mathbb{R}_3} \frac{1}{4\pi|x-y|}\Delta_y V(y)\, dy \Rightarrow$$

$$\Rightarrow \lim_{\varepsilon\downarrow 0^+} \int_{\varepsilon \le |r| \le m} \frac{\Delta V}{4\pi r} = \lim_{\varepsilon \downarrow 0^+}\Bigg[-\int_{\varepsilon\le r\le m} \overbrace{\Delta\left(\frac{1}{4\pi r}\right)}^{=\delta(x) \to 0} V + \overbrace{\int_{|r|=\varepsilon} \frac{1}{4\pi r}\frac{\partial V}{\partial r}}^{\to 0} - \overbrace{\int_{r=\varepsilon} \frac{\partial}{\partial r}\left(\frac{1}{4\pi r}\right) V}^{= \bar{V}_\varepsilon \xrightarrow{\varepsilon\to 0} V(0)} \Bigg] = V(0)$$

GREEN'S FUNCTIONS II. (Laplace/Poisson)

- green's function for Dirichlet: $\Delta v = f(x)$ $x \in \Omega$, $v = g(x)$ $x \in \partial\Omega$

$$G(x,\xi) : \begin{cases} \bullet \text{ continuous for } x \in \bar{\Omega} \setminus \{\xi\} \\ \bullet \text{ harmonic for } x \in \Omega \setminus \{\xi\} \\ \bullet = 0 \text{ on } x \in \partial\Omega \\ \bullet \Delta_x G(x,\xi) = \delta(x-\xi) \end{cases}$$

$$\Rightarrow v(\bar{\xi}) = \underbrace{\int_\Omega G(\bar{x},\bar{\xi}) f(\bar{x}) d\bar{x}}_{\sim \text{fundamental sol.}} + \int_{\partial\Omega} \underbrace{\frac{\partial G}{\partial n}(\bar{x},\bar{\xi})\, g(\bar{x})}_{\text{boundary term}} d\bar{x}$$

- method of odd reflections – 2D: $\bar{x} = (x,y)$, $\bar{\xi} = (\xi,\eta)$

known: $G_1 \equiv \frac{1}{2\pi} \log|(x,y)-(\xi,\eta)|$ No forcing else where this free

$$\Delta_{(x,y)} G_1 = \delta((x,y)-(\xi,\eta))$$

in free space: fundamental sol. of Laplace $= \frac{1}{2\pi}\log|(x,y)|$

☆ upper half-space $\Delta v = f$, $y>0$; $v = g$, $y=0$

$G_2 = \frac{1}{2\pi}\log|(x,y)-(\xi,-\eta)|$ ✓ solves Laplace in upper half-plane

✓ $\Delta G = \delta$

✓ 0 everywhere on boundary

$G \equiv G_1 - G_2$ is also 0 on boundary (both G_1 and G_2 are)

$$\Rightarrow \cancel{G = \frac{1}{2\pi}\log} \quad G = \frac{1}{4\pi}\log\left[\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2}\right]$$

☆ disc